

## Exercise 6

Repeat Exercise 5(a) for the data  $u(x, 0) = e^{-x^2}$ ,  $\frac{\partial u}{\partial t}(x, 0) = 0$ ,  $-\infty < x < \infty$ .

### Solution

The aim is to solve the wave equation on the whole line for all time subject to initial conditions.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad -\infty < t < \infty$$

$$u(x, 0) = f(x) = e^{-x^2}$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x) = 0$$

Start with the general solution of the wave equation.

$$u(x, t) = F(x + ct) + G(x - ct)$$

Differentiate it with respect to  $t$ .

$$\frac{\partial u}{\partial t} = F'(x+ct) \cdot \frac{\partial}{\partial t}(x+ct) + G'(x-ct) \cdot \frac{\partial}{\partial t}(x-ct) = F'(x+ct) \cdot (c) + G'(x-ct) \cdot (-c) = cF'(x+ct) - cG'(x-ct)$$

Now apply the given initial conditions.

$$u(x, 0) = F(x) + G(x) = f(x)$$

$$\frac{\partial u}{\partial t}(x, 0) = cF'(x) - cG'(x) = g(x)$$

This is a system of two equations with two unknowns,  $F$  and  $G$ , that can be solved for. Differentiate both sides of the first equation.

$$\begin{cases} F'(x) + G'(x) = f'(x) \\ cF'(x) - cG'(x) = g(x) \end{cases}$$

Multiply both sides of the first equation by  $c$ .

$$\begin{cases} cF'(x) + cG'(x) = cf'(x) \\ cF'(x) - cG'(x) = g(x) \end{cases}$$

Adding the respective sides of these equations eliminates  $G$  and gives

$$2cF'(x) = cf'(x) + g(x).$$

Divide both sides by  $2c$ .

$$F'(x) = \frac{1}{2}f'(x) + \frac{1}{2c}g(x)$$

Integrate both sides with respect to  $x$ .

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2c} \int^x g(s) ds + C_1$$

Subtracting the respective sides of these equations instead eliminates  $F$  and gives

$$2cG'(x) = cf'(x) - g(x).$$

Divide both sides by  $2c$ .

$$G'(x) = \frac{1}{2}f'(x) - \frac{1}{2c}g(x)$$

Integrate both sides with respect to  $x$ .

$$\begin{aligned} G(x) &= \frac{1}{2}f(x) - \frac{1}{2c} \int^x g(s) ds + C_2 \\ &= \frac{1}{2}f(x) + \frac{1}{2c} \int_x g(s) ds + C_2 \end{aligned}$$

Now that  $F$  and  $G$  are known, the solution to the initial value problem can be written.

$$\begin{aligned} u(x, t) &= F(x + ct) + G(x - ct) \\ &= \left[ \frac{1}{2}f(x + ct) + \frac{1}{2c} \int^{x+ct} g(s) ds + C_1 \right] + \left[ \frac{1}{2}f(x - ct) + \frac{1}{2c} \int_{x-ct} g(s) ds + C_2 \right] \\ &= \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds + C_3 \end{aligned}$$

The integration constant is set to zero to satisfy  $u(x, 0) = f(x)$ .

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

In this exercise

$$f(x) = e^{-x^2} \quad \text{and} \quad g(x) = 0,$$

so

$$\begin{aligned} u(x, t) &= \frac{1}{2} \left[ e^{-(x+ct)^2} + e^{-(x-ct)^2} \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} (0) ds \\ &= \frac{1}{2} \left[ e^{-(x+ct)^2} + e^{-(x-ct)^2} \right]. \end{aligned}$$

Below is a plot of  $u(x, t)$  versus  $x$  over  $-15 < x < 15$  for  $t = 0, 1, 2, 4, 6, 8$  with  $c = 1$ .

